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ISOMETRIES OF CSL ALGEBRAS

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ABSTRACT. We show that every Jordan isomorphism of CSL algebras, whose restriction to the diagonal of the algebra is a selfadjoint map, is the sum of an isomorphism and an anti-isomorphism.

It follows that every surjective linear isometry of CSL algebras is the sum of an isomorphism and an anti-isomorphism, followed by a unitary multiplication.

1. INTRODUCTION

In [K], R. Kadison proved that every linear isometry of one C^*-algebra onto another is given by a Jordan $*$-isomorphism followed by a unitary multiplication. (Here a linear map φ from a C^*-algebra $\mathscr{B}_1$ into a C^*-algebra $\mathscr{B}_2$ is called a *Jordan $*$-isomorphism* if it is one-to-one, surjective, $\varphi(x^*) = \varphi(x)^*$ and $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$ for all $x, y \in \mathscr{B}_1$.) Kadison also proved that a Jordan $*$-isomorphism from a von Neumann algebra $\mathscr{B}_1$ onto a von Neumann algebra $\mathscr{B}_2$ can be decomposed into the sum of a $*$-isomorphism and of a $*$-anti-isomorphism by a central projection.

For nonselfadjoint algebras the following general result was proved in [AS].

Theorem 1.1. *Let $\mathscr{U} \subseteq B(H)$ and $\mathscr{B} \subseteq B(K)$ be unital norm closed subalgebras, and let $\varphi: \mathscr{U} \to \mathscr{B}$ be a surjective linear isometry. Then*

(1) $\varphi(\mathscr{U} \cap \mathscr{U}^*) = \mathscr{B} \cap \mathscr{B}^*$.

(2) $\varphi(xy^*z + zy^*x) = \varphi(x)\varphi(y)^*\varphi(z) + \varphi(z)\varphi(y)^*\varphi(x)$ *for every* x, z *in* $\mathscr{U}$ *and* y *in* $\mathscr{U} \cap \mathscr{U}^*$.

(3) $U = \varphi(I)$ *is a unitary operator in* $\mathscr{B} \cap \mathscr{B}^*$.

If, moreover, $\varphi(I) = I$, *then*

(4) $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$, $x, y \in \mathscr{U}$.

(5) $\varphi(x^*) = \varphi(x^*)$ *for* $x \in \mathscr{U} \cap \mathscr{U}^*$.

Note that (2) shows that φ preserves the partial triple product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$, $y \in \mathscr{U} \cap \mathscr{U}^*$, $x, z \in \mathscr{U}$. We shall therefore refer to a surjective, one-to-one linear map that satisfies (1)–(3) as a *partial triple $*$-isomorphism.*

Also, a surjective one-to-one linear map $\varphi: \mathscr{U} \to \mathscr{B}$ will be called *Jordan partial $*$-isomorphism* if it maps I to I and both it and its inverse satisfy

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properties (4) and (5) above. It follows from Theorem 1.1 that every isometry between unital normed closed operator algebras is given by a Jordan partial $*$-isomorphism followed by a unitary multiplication (thus extending Kadison's result to nonselfadjoint algebras).

The main result of the present paper (Theorem 2.15) is that, when $\mathscr{U}$ and $\mathscr{B}$ are reflexive operator algebras with commutative subspace lattices (called CSL algebras), then every partial Jordan $*$-isomorphism can be decomposed into the sum of an isomorphism and an anti-isomorphism by a projection in the center of $\mathscr{U}$.

In particular, every isometry from CSL algebra onto another is such a sum followed by a unitary multiplication. For completely distributive CSL algebras a more concrete result was proved by R. Moore and T. Trent in [MT2].

In order to state their result we shall first set some notation and terminology. A Hilbert space H would be assumed to be separable, an operator on H would be assumed to be bounded and a projection is assumed to be orthogonal. A lattice $\mathscr{L}$ of projections is a strongly closed collection of projections that is closed under the usual lattice operations $\vee$ and $\wedge$ and contains 0 and I. In this paper we will deal only with commutative lattices (in which the projections commute pairwise). Such a lattice is called a *CSL*. A *nest* is a linearly ordered lattice. If $\mathscr{L}$ is a lattice we write $\operatorname{Alg}\mathscr{L}$ for the collection of operators in $B(H)$ which leave invariant the ranges of all of the projections in $\mathscr{L}$, i.e.

$$\operatorname{Alg}\mathscr{L} = \{T \in B(H) \colon (I - N)TN = 0,\ N \in \mathscr{L}\}.$$

$\operatorname{Alg}\mathscr{L}$ is a weakly closed subalgebras of $B(H)$, containing I. If $\mathscr{A}$ is a subalgebra of $B(H)$ we write

$\mathscr{L}at_{\mathscr{A}} = \{N \colon N$ is a projection, $(I - N)TN = 0$ for all $T \in \mathscr{A}\}$. If $\mathscr{L}$ is commutative, then $\mathscr{L}at(\operatorname{Alg}\mathscr{L}) = \mathscr{L}$ [A, Theorem 1.6.3]. For a projection E we write $E^{\perp} = I - E$ and for a lattice $\mathscr{L}$ we write $\mathscr{L}^{\perp} = \{N^{\perp} \colon N \in \mathscr{L}\}$. Note also that $\operatorname{Alg}(\mathscr{L}^{\perp}) = (\operatorname{Alg}\mathscr{L})^*$ and $(\operatorname{Alg}\mathscr{L})^* \cap \operatorname{Alg}\mathscr{L} = \mathscr{L}'$.

A CSL $\mathscr{L}$ is said to be *completely distributive* if it satisfies a certain lattice-theoretic condition (see [D, Chapter 23]). An alternative characterization of completely distributive CSL was proved by Laurie and Longstaff [LL]. They showed that $\mathscr{L}$ is completely distributive if and only if the linear span of the rank one operators in $\operatorname{Alg}\mathscr{L}$ is σ-weakly dense in $\operatorname{Alg}\mathscr{L}$.

The main result of [MT2] is the following.

Theorem 1.2 [MT2, Theorem 2.1]. *Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be completely distributive CSL's on a Hilbert space H. Let $\theta\colon \operatorname{Alg}\mathscr{L}_1 \to \operatorname{Alg}\mathscr{L}_2$ be a linear surjective isometry. Let $U = \theta(I)$ and write $\varphi(T) = U^*\theta(T)$. Then there exist projections $E_1 \in \mathscr{L}_1 \cap \mathscr{L}_1^{\perp}$, $E_2 \in \mathscr{L}_2 \cap \mathscr{L}_2^{\perp}$ and an involution J such that*

(1) *φ restricted to $(\operatorname{Alg}\mathscr{L}_1)E_1$, is implemented by a unitary operator V (i.e. $\varphi(T) = V^*TV$) such that $N \mapsto V^*NV$ is an order isomorphism of $\mathscr{L}_1E_1$ onto $\mathscr{L}_2E_2$.*

(2) *For φ, restricted to $(\operatorname{Alg}\mathscr{L}_1)E_1^{\perp}$, there is a unitary operator W such that $\varphi(T) = W^*JTJW$ and the map $N \mapsto W^*JNJW$ is an order isomorphism from $\mathscr{L}_1E_1^{\perp}$ onto $\mathscr{L}_2^{\perp}E_2^{\perp}$.*

The proof of this result in [MT2] uses heavily the fact that there are many rank one operators in $\operatorname{Alg}\mathscr{L}$. In this paper we deal with general CSL lattices and, in the general case, $\operatorname{Alg}\mathscr{L}$ might contain no rank one operators. Therefore the methods are completely different. As we remark at the end of the paper, our main result (Theorem 2.15) can be combined with [DP, Theorem 2.1] to yield an alternative proof for Theorem 1.2.

For the special case where $\mathscr{L}_1$ and $\mathscr{L}_2$ are nests Theorem 1.2 was proved, independently (and using different methods), in [AS and MT1]. In this case either $E_1 = 0$ or $E_1 = I$ (as $\mathscr{L} \cap \mathscr{L}^{\perp} = \{0, I\}$).

2. Jordan partial $*$-isomorphisms

We now fix two CSL's $\mathscr{L}$ and $\mathscr{L}_1$ and φ: $\operatorname{Alg}\mathscr{L} \to \operatorname{Alg}\mathscr{L}_1$, a *Jordan partial $*$-isomorphism*; i.e. φ is linear, one-to-one, surjective, $\varphi(I) = I$, $\varphi(x^*) = \varphi(x)^*$ for $x \in \operatorname{Alg}\mathscr{L} \cap (\operatorname{Alg}\mathscr{L})^*$, $\varphi(xy + yx) = \varphi(x)\varphi(y) + \varphi(y)\varphi(x)$ for x, $y \in \operatorname{Alg}\mathscr{L}$ and its inverse also satisfies these properties. We have the following:

(I) $\varphi(\operatorname{Alg}\mathscr{L} \cap (\operatorname{Alg}\mathscr{L})^*) = \operatorname{Alg}\mathscr{L}_1 \cap (\operatorname{Alg}\mathscr{L}_1)^*$; i.e. $\varphi(\mathscr{L}') = \mathscr{L}_1'$.

(II) As φ preserves commutativity, $\varphi(\mathscr{L}'') = \mathscr{L}_1''$. Thus φ, restricted to $\mathscr{L}''$ (which is an abelian von Neumann algebra), is a $*$-isomorphism. In particular, for every collection of projections $\{E_\alpha\} \subseteq \mathscr{L}''$, $\varphi(\bigvee E_\alpha) = \bigvee \varphi(E_\alpha)$ and $\varphi(\bigwedge E_\alpha) = \bigwedge \varphi(E_\alpha)$.

(III) For all R, S, $T \in \operatorname{Alg}\mathscr{L}$ we have $\varphi(RST + TSR) = \varphi(R)\varphi(S)\varphi(T) + \varphi(T)\varphi(S)\varphi(R)$ [AS, Corollary 2.11].

We write $\mathscr{A}$ for $\operatorname{Alg}\mathscr{L}$ and for $\mathscr{A}_1$ for $\operatorname{Alg}\mathscr{L}_1$. For a subset $S \subseteq H$ we write [S] for the closed linear subspace spanned by S.

Lemma 2.1. *The following are equivalent for a projection $E \in \mathscr{L}''$.*

(1) $E = E_1 - E_2$ *for some* $E_i \in \mathscr{L}$, $i = 1, 2$.

(2) *For every* T, S *in* $\operatorname{alg}\mathscr{L}$, $ETESE = ETSE$.

(3) *For every* T, S *in* $\operatorname{alg}\mathscr{L}$,

$$ETESE + ESETE = ETSE + ESTE.$$

Proof. (1) $\Rightarrow$ (2). For every T, S in $\operatorname{alg}\mathscr{L}$ and $E = E_1 - E_2$, E_1, $E_2 \in \mathscr{L}$ we have

$$ET = E_1(1 - E_2)T = E_1(1 - E_2)T(1 - E_2)$$

and $SE = SE_1(1 - E_2) = E_1SE$; hence $ETSE = ETESE$. (2) $\Rightarrow$ (3) is obvious.

(2) $\Rightarrow$ (1). Assume (2) holds. Write E_1 for the orthogonal projection onto $[\mathscr{A}E(H)]$. Then $E_1 \in \mathscr{L}$. It is left to show that $E_1 - E \in \mathscr{L}$. Fix $T \in \mathscr{A}$ and x in $(E_1 - E)(H)$. Suppose $x = SEy$ for some $S \in \mathscr{A}$ (and $EX = 0$). Clearly $Tx = TSEy \in E_1(H)$ and also $ETx = ETSEy = ETESEy = ETEx = 0$. Hence $Tx \in (E_1 - E)(H)$. As vectors of the form SEy are dense in $(E_1 - E)(H)$ we see that $Tx \in (E_1 - E)(H)$ for all $x \in (E_1 - E)(H)$.

(3) $\Rightarrow$ (2). Let $\{E_i\}_{i=1}^{\infty}$ be a countable subset of $\mathscr{L}$ that is strongly dense in $\mathscr{L}$ and such that $E_1 = 0$, $E_2 = I$. As in [A, Proof Theorem 2.2.3] we fix $n \geq 1$ and for each n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i = +1$ or -1 we define $E^\alpha = E_1^{\alpha_1} E_1^{\alpha_2} \cdots E_n^{\alpha_n}$ where $E_i^1 = E_i$ and $E_i^{-1} = I - E_i$.

As α runs over all n-tuples E^α runs over the atoms of the Boolean algebra generated by $\{E_1, \ldots, E_n\}$. Also, if $\alpha \neq \beta$, $E^\alpha E^\beta = 0$. We assume here that $ETESE + ESETE = ETSE + ESTE$ for all T, $S \in \mathscr{A}$. Apply this to $E^\alpha T$ and SE^β for some $\alpha \neq \beta$ to get $EE^\alpha TSE^\beta E = EE^\alpha TESE^\beta E$ (as $SE^\beta E^\alpha T = SE^\beta EE^\alpha T = 0$). Write $K = ETSE - ETESE$. Then $E^\alpha KE^\beta = 0$

for all $\alpha \neq \beta$. As $\sum E^\alpha = I$ we get

$$K = \sum_\alpha E^\alpha K E^\alpha \quad \text{and} \quad K \in \{E_1, \ldots, E_n\}'.$$

We have, for $1 \leq i \leq n$,

$$KE_i = ET(1-E)SEE_i = ET(1-E)E_iSE_iE$$

and

$$\begin{aligned} K(I-E_i) &= (I-E_i)K(I-E_i) = (1-E_i)ET(I-E)SE(I-E_i) \\ &= ET(I-E)(I-E_i)S(I-E_i)E. \end{aligned}$$

Using this repeatedly we get

$$KE^\alpha = ET(1-E)E^\alpha SE^\alpha E$$

for all α. Hence $K = \sum KE^\alpha = ET(1-E)(\sum_\alpha E^\alpha SE^\alpha)E$. Write $S_n = \sum_\alpha E^\alpha SE^\alpha$. Then $S_n \in \{E_1, \ldots, E_n\}'$ and $\|S_n\| \leq \|S\|$. Hence there is a weakly convergent subsequence $S_{n_k} \to S_0$. Clearly $S_0 \in \mathscr{L}'$ and thus $K = ET(1-E)S_0E$. As $S_0 \in \mathscr{L}'$, $S_0E = ES_0$ and $K = 0$. This proves (2). □

Corollary 2.2. *If $E = E_1 - E_2$, $E_i \in \mathscr{L}$, then there are projections Q_1, Q_2, in $\mathscr{L}_1$, such that $\varphi(E) = Q_1 - Q_2$.*

Proof. With E as above we have

$$ETESE + ESETE = ETSE + ESTE$$

for every T, $S \in \mathscr{A}$. Since φ preserves Jordan products and $\varphi(ERE) = \varphi(E)\varphi(R)\varphi(E)$ for every $R \in \mathscr{A}$, we have

$$\begin{aligned} \varphi(ETESE + ESETE) &= \varphi(ETE)\varphi(ESE) + \varphi(ESE)\varphi(ETE) \\ &= \varphi(E)\varphi(T)\varphi(E)\varphi(S)\varphi(E) + \varphi(E)\varphi(S)\varphi(E)\varphi(T)\varphi(E). \end{aligned}$$

But this is equal to

$$\begin{aligned} \varphi(ETSE + ESTE) &= \varphi(E)\varphi(TS + ST)\varphi(E) \\ &= \varphi(E)\varphi(S)\varphi(T)\varphi(E) + \varphi(E)\varphi(T)\varphi(S)\varphi(E). \end{aligned}$$

As φ is surjective we can use Lemma 2.1 to complete the proof. □

A projection E in $\mathscr{L}''$ that can be written as $E_1 - E_2$ for some (not necessarily unique) E_1 and E_2 in $\mathscr{L}$ is called an *interval*.

Now fix $N \in \mathscr{L}$. Write $E = \varphi(N)$. By Corollary 2.2, $E = E_1 - E_2$ for some E_1, $E_2 \in \mathscr{L}_1$. (Note that the choice of E_1 and E_2 is not unique. We choose one possible pair.) Applying Corollary 2.2 to φ^{-1} we can find intervals Q and P such that $\varphi(Q) = I - E_1$ and $\varphi(P) = E_2$. As $E_2 + (I - E_1) = I - E$, $P + Q = I - N$. Also, as $E_2(I - E_1) = 0$, $PQ = 0$.

Lemma 2.3. *With N fixed in $\mathscr{L}$ and E, E_1, E_2, P, Q as above we have*

$$\varphi(NTP) = E_2\varphi(T)E \quad \text{and} \quad \varphi(NTQ) = E\varphi(T)(I - E_1)$$

for all $T \in \mathscr{A}$.

Proof. For $T \in \mathscr{A}$ we have $QTN = 0$ (as $N \in \mathscr{L}$ and $Q \leq I - N$) and $(I - E_1)\varphi(T)E = (I - E_1)\varphi(T)E_1E = 0$ (as $E_1 \in \mathscr{L}_1$). Using property (III) of φ (see the beginning of this section) we now have,

$$\begin{aligned} \varphi(NTQ) = \varphi(NTQ + QTN) &= E\varphi(T)(I - E_1) + (I - E_1)\varphi(T)E \\ &= E\varphi(T)(I - E_1). \end{aligned}$$

The proof for NTP is similar and is omitted. □

Lemma 2.4. *Write* F_1, F_2 *for the projection onto* $[N\mathcal{A}P(H)]$ *and* $[N\mathcal{A}Q(H)]$ *respectively. Then* $\varphi(F_1)$ *and* $\varphi(F_2)$ *are the projections onto* $[E\mathcal{A}_1^*E_2(H)]$ *and* $[E\mathcal{A}_1(I-E_1)(H)]$ *respectively.*

For every $T \in \mathcal{A}$ we have $F_1NTP = NTP$. Hence $NTP = F_1NTP + NTPF_1$ (as $F_1 \leq N \leq I-P$) and, thus, $E_2\varphi(T)E = \varphi(NTP) = \varphi(F_1)E_2\varphi(T)E + E_2\varphi(T)E\varphi(F_1)$. But $\varphi(F_1)E_2 = \varphi(F_1P) = 0$; hence for every $T \in \mathcal{A}$,

$$E_2\varphi(T)E = E_2\varphi(T)E\varphi(F_1)$$

and for every $S \in \mathcal{A}_1^*$,

$$ESE_2 = \varphi(F_1)ESE_2.$$

Hence $\varphi(F_1)$ is larger or equal to the projection onto $[E\mathcal{A}_1^*E_2(H)]$. If we write F_1' for the latter projection we have

$$ET^*E_2 = F_1'ET^*E_2$$

for every $T \in \mathcal{A}_1$. We have $E_2TE = (ET^*E_2)^* = (F_1'ET^*E_2)^* = E_2TEF_1' = E_2TEF_1' + F_1'E_2TE$ (as $F_1' \leq E \leq I - E_2$). Hence, for $T \in \mathcal{A}_1$, $N\varphi^{-1}(T)P = \varphi^{-1}(E_2TE) = \varphi^{-1}(E_2TE)\varphi^{-1}(F_1') + \varphi^{-1}(F_1')\varphi(E_2TE) = N\varphi^{-1}(T)P\varphi^{-1}(F_1') + \varphi^{-1}(F_1')N\varphi^{-1}(T)P = \varphi^{-1}(F_1')N\varphi^{-1}(T)P$ as $P\varphi^{-1}(F_1') = \varphi^{-1}(E_2F_1') = 0$. Hence $\varphi^{-1}(F_1') \geq F_1$; i.e. $F_1' \geq \varphi(F_1)$ proving that $\varphi(F_1)$ is equal to F_1'. The proof for $[N\mathcal{A}Q(H)]$ is similar and is omitted. □

Lemma 2.5. *With the notation of Lemma* 2.4, $F_1F_2 = 0$.

Proof. For every T, S in $\mathcal{A}$ we have $0 = NTPNSQ + NSQNTP$ as $PN = QN = 0$. Hence

$$\begin{aligned} 0 &= \varphi(NTPNSQ + NSQNTP) = \varphi(NTP)\varphi(NSQ) + \varphi(NSQ)\varphi(NTP) \\ &= E_2\varphi(T)EE\varphi(S)(I-E_1) + E\varphi(S)(I-E_1)E_2\varphi(T)E \\ &= E_2\varphi(T)E\varphi(S)(I-E_1) \end{aligned}$$

as $(I-E_1)E_2 = 0$. Hence $E_2\mathcal{A}_1E\mathcal{A}_1(I-E_1) = \{0\}$. From Lemma 2.4 it follows that $E_2\mathcal{A}_1E\varphi(F_2) = \{0\}$; hence $\varphi(F_2)E\mathcal{A}_1^*E_2 = \{0\}$. Using Lemma 2.4 again we get $\varphi(F_2)\varphi(F_1) = 0$ and, thus, $F_1F_2 = 0$.

Note here that both F_1 and F_2 are in $\mathcal{L}$ (to see that $F_1 \in \mathcal{L}$ just observe that for T, $S \in \mathcal{A}$, $TNSP = NTNSP$; so that $TF_1 = F_1TF_1$. The proof for F_2 is similar). Also $F_1 + F_2 \leq N$.

For the projection $N \in \mathcal{L}$ above we associated projections E, E_1, E_2, Q, P, F_1, F_2 etc. This can be done for every projection $L \in \mathcal{L}$ and then we shall write $E(L)$, $E_1(L)$, $E_2(L)$, etc.

Lemma 2.6. *Fix* $N \in \mathcal{L}$ *as above and let* F_1, F_2 *be the projections defined above. Then, for every projection* $M \in \mathcal{L}$,

(1) $I - \varphi(MF_1) \in \mathcal{L}_1$,
(2) $\varphi(MF_2) \in \mathcal{L}_1$.

Proof. We shall prove (1). The proof of (2) is similar. Note that from Lemma 2.4 we know:

(i) $E_2\mathcal{A}_1E(I-\varphi(F_1)) = \{0\}$,
(ii) $(I-\varphi(F_2))E\mathcal{A}_1(I-E_1) = \{0\}$, and from Lemma 2.5,
(iii) $F_1F_2 = 0$.

Now (ii) and (iii) imply that $\varphi(MF_1)\mathscr{A}_1(I-E_1)=\{0\}$. To prove (1) we have to show that

$$\varphi(MF_1)\mathscr{A}_1(I-\varphi(MF_1))=\{0\}.$$

Write $G=MF_1$. Then $G\in\mathscr{L}$ and $E(G)$, $E_1(G)$, etc. are well defined. Applying (ii)–(iii) to G, in place of N, we get

(i′) $E_2(G)\mathscr{A}_1E(G)(I-\varphi(F_1(G)))=\{0\}$,

(ii′) $(I-\varphi(F_2(G)))E(G)\mathscr{A}_1(I-E_1(G))=\{0\}$,

(iii′) $F_1(G)F_2(G)=0$.

As $E(G)=\varphi(G)\leq\varphi(N)=E\leq I-E_2$ and $E(G)=E_1(G)-E_2(G)$, it is easy to check that $E(G)=E_2\vee E_1(G)-E_2\vee E_2(G)$; hence, replacing $E_i(G)$, $i=1,2$, by $E_i(G)\vee E_2$, we can assume that $E_i(G)\geq E_2$. (Recall that the choice of $E_1(G)$ and $E_2(G)$ was arbitrary.)

We now get, from (i′), $E_2\mathscr{A}_1E(G)(I-\varphi(F_1(G)))=\{0\}$, i.e.

$$(1-\varphi(F_1(G)))E(G)\mathscr{A}_1{}^*E_2=\{0\}.$$

But $E(G)=\varphi(G)=\varphi(F_1)\varphi(M)$. Hence $(1-\varphi(F_1(G)))\varphi(M)\varphi(F_1)\mathscr{A}_1{}^*E_2=\{0\}$. As $\varphi(F_1)$ is the projection onto $[E\mathscr{A}_1{}^*E_2(H)]$ and $E\geq\varphi(F_1)$, it is the projection onto $[\varphi(F_1)\mathscr{A}_1{}^*E_2(H)]$. We therefore have

$$0=(I-\varphi(F_1(G)))\varphi(M)\varphi(F_1)=(I-\varphi(F_1(G)))\varphi(G).$$

It follows that $G(I-F_1(G))=0$. But $F_1(G)\leq G$; hence $G=F_1(G)$. As $F_2(G)\leq G-F_1(G)$, $F_2(G)=0$ and, using (ii′) we get

$$(*)\qquad E(G)\mathscr{A}_1(I-E_1(G))=\{0\}.$$

Also $E(G)\mathscr{A}_1E_2(G)=\{0\}$ as $E_2(G)\in\mathscr{L}_1$ and $E_2(G)E(G)=0$. Combining this with $(*)$ we have

$$E(G)\mathscr{A}_1(I-E(G))=\{0\}.$$

Hence $\varphi(MF_1)=\varphi(G)=E(G)\in\mathscr{L}_1^{\perp}$. □

We shall now write $P_0=\bigvee\{P\in\mathscr{L}\cap\mathscr{L}^{\perp};\ \mathscr{A}P\subseteq\mathscr{L}'\}$ and note that $\mathscr{A}P_0\subseteq\mathscr{L}'$.

Lemma 2.7. *For N and M in $\mathscr{L}$, $F_1(N)F_2(M)\leq P_0$.*

Proof. Write $L=F_1(N)F_2(M)\in\mathscr{L}$. As $L\leq F_1(N)$ and $L\leq F_2(M)$, Lemma 2.6 implies that $\varphi(L)\in\mathscr{L}_1\cap\mathscr{L}_1^{\perp}$. Note that $\mathscr{L}\cap\mathscr{L}^{\perp}$ is the set of all projections in $\mathscr{A}'$. Thus $\varphi(\mathscr{L}\cap\mathscr{L}^{\perp})=\mathscr{L}_1\cap\mathscr{L}_1^{\perp}$ and thus $L\in\mathscr{L}\cap\mathscr{L}^{\perp}$. In fact, for every $N_1\in\mathscr{L}$ $N_1L\leq L$ and the same argument shows that $N_1L\in\mathscr{L}\cap\mathscr{L}^{\perp}$. Hence $\mathscr{L}L\subseteq\mathscr{L}\cap\mathscr{L}^{\perp}$ from which it easily follows that $\mathscr{A}L\subseteq\mathscr{L}'$. Hence $L\leq P_0$. □

We now write $F_+=\bigvee\{F_2(M)\colon M\in\mathscr{L}\}$ and $F_-=\bigvee\{F_1(M)\colon M\in\mathscr{L}\}$. If $N\in\mathscr{L}$ and $N\leq F_+$ then $N=\bigvee\{NF_2(M)\colon M\in\mathscr{L}\}$ and $\varphi(N)=\bigvee\{\varphi(NF_2(M))\colon M\in\mathscr{L}\}\in\mathscr{L}_1$ (by Lemma 2.6). Similarly whenever $N\in\mathscr{L}$ and $N\leq F_-$, $\varphi(N)\in\mathscr{L}_1^{\perp}$. Note also that $F_+F_-\leq P_0$ (by Lemma 2.7). Suppose that $P_0=0$ so that $F_+F_-=0$.

Write $A=F_++F_-\in\mathscr{L}$ and note that $\varphi(I-A)=(I-\varphi(F_+))-\varphi(F_-)$ and $I-\varphi(F_+)$, $\varphi(F_-)\in\mathscr{L}_1^{\perp}$.

We can thus apply Lemma 2.5 and Lemma 2.6 with $\mathscr{A}^*$, $\mathscr{A}_1{}^*$ replacing $\mathscr{A}$ and $\mathscr{A}_1$, $\mathscr{L}^{\perp}$ and $\mathscr{L}_1^{\perp}$ replacing $\mathscr{L}$ and $\mathscr{L}_1$, φ^* (defined by $\varphi^*(T^*)=\varphi(T)^*$) replacing φ and $I-A$, F_+ and F_- replacing N, Q and P respectively to get the following.

Lemma 2.8. *Suppose* $P_0 = 0$ *and let* A_+ *and* A_- *be the projections onto* $[(I-A)\mathscr{A}^*F_+(H)]$ *and* $[(I-A)\mathscr{A}^*F_-(H)]$ *respectively. Then*

(1) $A_+A_- = 0$.

(2) *For every projection* $M \in \mathscr{L}$, $\varphi((I-M)A_-) \in \mathscr{L}_1$. □

Lemma 2.9. $(I-A)\mathscr{A} \subseteq \mathscr{L}'$.

Proof. Fix $N \in \mathscr{L}$. Recall that $F_1(N)$ is the projection onto $[NAP(H)]$, $F_2(N)$ is the projection onto $[N\mathscr{A}Q(H)]$, $Q+P = I-N$ and $F_1(N)+F_2(N) \leq A$. Hence

$$(I-A)N\mathscr{A}(I-N) = \{0\}.$$

also $(I-N)AN = \{0\}$ (as $N \in \mathscr{L}$). We therefore have

$$(I-N)(I-A)\mathscr{A}N = \{0\} = N(I-A)\mathscr{A}(I-N).$$

Hence $(I-A)\mathscr{A} \subseteq \mathscr{L}'$. □

Finally, write $E_+ = F_+ + A_+$ and $E_- = F_- + A_-$.

Lemma 2.10. *Suppose* $P_0 = 0$. *Then*

(1) E_+ *and* E_- *are in* $\mathscr{L} \cap \mathscr{L}^\perp$.

(2) $E_+ + E_- = I$.

(3) *For every* $M \in \mathscr{L}$, $\varphi(ME_+) \in \mathscr{L}_1$ *and* $\varphi(ME_-) \in \mathscr{L}_1^\perp$.

Proof. First we show that $E_+ \in \mathscr{L}$. As $F_+ \in \mathscr{L}$ it suffices to show that $(I-E_+)\mathscr{A}A_+ = 0$. As $I - E_+ = F_- + (I-A_+)(I-A)$, we shall show

(i) $F_-\mathscr{A}A_+ = \{0\}$; and

(ii) $(I-A_+)(I-A)\mathscr{A}A_+ = \{0\}$.

For (i) note that $A_+A_- = 0$ and A_- is the projection on $[(I-A)\mathscr{A}^*F_-(H)]$; hence $A_+(I-A)\mathscr{A}^*F_- = \{0\}$ and, thus, $F_-\mathscr{A}(I-A)A_+ = 0$. As $A_+ = (I-A)A_+$, this proves (i). (ii) follows immediately from the fact that $(I-A)\mathscr{A} \subseteq \mathscr{L}'$ (Lemma 2.9). Hence $E_+ \in \mathscr{L}$. The proof that $E_- \in \mathscr{L}$ is almost identical and is omitted.

Now write $A_0 = I - E_+ - E_- \leq I - F_+ - F_- = I - A$. As $(I-A)\mathscr{A} \subseteq \mathscr{L}'$ (Lemma 2.9), also $A_0\mathscr{A} \subseteq \mathscr{L}'$. Note also that, as $A_0A_+ = A_0A_- = 0$ we have, $A_0\mathscr{A}^*F_+ = A_0\mathscr{A}^*F_- = \{0\}$; hence $A_0\mathscr{A}^*A = \{0\}$ and, thus, $A\mathscr{A}A_0 = \{0\}$. As $(I-A)\mathscr{A} \subseteq \mathscr{L}'$, we have $(I-A)\mathscr{A}A_0 \subseteq \mathscr{L}'$. Therefore $\mathscr{A}A_0 = A\mathscr{A}A_0 + (I-A)\mathscr{A}A_0 \subseteq \mathscr{L}'$. Hence we have $\mathscr{A}A_0 \subseteq \mathscr{L}'$ and $A_0\mathscr{A} \subseteq \mathscr{L}'$ and this implies that $A_0 \in \mathscr{A}'$ (as $(I-A_0)\mathscr{A}A_0 \subseteq (I-A_0)\mathscr{L}'A_0 = \{0\}$ and, similarly $A_0\mathscr{A}(I-A_0) = \{0\}$); i.e. $A_0 \leq P_0 = 0$.

This shows that $E_+ + E_- = I$ and, as E_+, $E_- \in \mathscr{L}$ we have E_+, $E_- \in \mathscr{L} \cap \mathscr{L}^\perp$. This proves parts (1) and (2). For (3) we fix $M \in \mathscr{L}$. We want to show that $\varphi(ME_+) \in \mathscr{L}_1$. As $E_+ \in \mathscr{L} \cap \mathscr{L}^\perp \subseteq \mathscr{A}'$ we can replace $\mathscr{L}$ by $\mathscr{L}E^+$ and $\mathscr{A}$ by $\mathscr{A}E_+$; i.e. we assume $E_+ = I$. Then $F_+ + A_+ = I$.

Note first that by Lemma 2.8(2) we know that $\varphi((I-M)A_+) \in \mathscr{L}_1^\perp$. Hence we have

$$(*)\quad \varphi((I-M)A_+)\mathscr{A}_1\varphi(MA_+) = \varphi((I-M)A_+)\mathscr{A}_1\varphi((I-M)A_+)\varphi(MA_+) = \{0\}.$$

For every $T \in \mathscr{A}$, $F_+(I-M)TMA_+ = 0$ (as $M \in \mathscr{L}$) and $MA_+T(I-M)F_+ = 0$ (as $F_+ \in \mathscr{L}$ and $A_+F_+ = 0$). Hence

$$\begin{aligned} 0 &= \varphi(F_+(I-M)TMA_+ + MA_+T(I-M)F_+) \\ &= \varphi(F_+(I-M))\varphi(T)(MA_+) + \varphi(MA_+)\varphi(T)\varphi((I-M)F_+). \end{aligned}$$

As $\varphi(F_+) \in \mathscr{L}$ and $\varphi(A_+)\varphi(F_+) = 0$, $\varphi(MA_+)\varphi(T)\varphi((I-M)F_+) = 0$. Hence $\varphi(F_+(I-M))\mathscr{A}_1\varphi(MA_+) = \{0\}$.

As $I - M = (A_+ + F_+)(I-M) = A_+(I-M) + F_+(I-M)$, combining this with $(*)$ we get

$$\varphi(I-M)\mathscr{A}_1\varphi(MA_+) = \{0\}.$$

As $\varphi(MF_+) \in \mathscr{L}_1$ (see the discussion following Lemma 2.7),

$$\varphi(I-M)\mathscr{A}_1\varphi(M) = \varphi(I-M)\mathscr{A}_1(\varphi(MA_+) + \varphi(MF_+)) = \{0\}.$$

Thus $\varphi(MA_+) \in \mathscr{L}_1$.

The proof that $\varphi(MA_-) \in \mathscr{L}^\perp$ is similar and is omitted. □

Proposition 2.11. *Let $\varphi: \mathscr{A} \to \mathscr{A}_1$ be a Jordan partial $*$-isomorphism of $\mathscr{A}$ onto $\mathscr{A}_1$. Then there are projections E_+, E_- and P_0 in $\mathscr{L} \cap \mathscr{L}^\perp$ such that $E_+ + E_- + P_0 = I$ and*

(1) *$AP_0 = \mathscr{L}'P_0$ (hence $\varphi(\mathscr{L}P_0) \subseteq \mathscr{L}_1 \cap \mathscr{L}_1^\perp$);*
(2) *for every $M \in \mathscr{L}$, $\varphi(ME_+) \in \mathscr{L}_1$ and, in fact, $\varphi(\mathscr{L}E_+) = \mathscr{L}_1\varphi(E_+)$; and*
(3) *for every $M \in \mathscr{L}$, $\varphi(ME_-) \in \mathscr{L}_1^\perp$ and, in fact $\varphi(\mathscr{L}E_-) = \mathscr{L}_1^\perp\varphi(E_-)$.*

Proof. Let $P_0 = \bigvee\{P \in \mathscr{L}\cap\mathscr{L}^\perp : \mathscr{A}P \subseteq \mathscr{L}'P\}$ (as above) and replace $\mathscr{A}$, $\mathscr{A}_1$, $\mathscr{L}$, $\mathscr{L}_1$ by $\mathscr{A}P_0^\perp$, $\mathscr{A}_1\varphi(P_0^\perp)$, $\mathscr{L}P_0^\perp$, $\mathscr{L}_1\varphi(P_0^\perp)$ respectively; i.e. assume $P_0 = 0$. Then let E_+ and E_- be the projections constructed above. By Lemma 2.10 we have $E_+ + E_- = I$. Part (1) now follows from the definition of P_0. From Lemma 2.10 we know that $\varphi(\mathscr{L}E_+) \subseteq \mathscr{L}_1\varphi(E_+)$ and $\varphi(\mathscr{L}E_-) \subseteq \mathscr{L}_1^\perp\varphi(E_-)$. Now write $P_0' = \bigvee\{P \in \mathscr{L}_1' \cap \mathscr{L}_1^\perp : \mathscr{A}_1P \subseteq \mathscr{L}_1'P\}$. It is clear that $\varphi(P_0) = P_0'$ and since we assume that $P_0 = 0$, also $P_0' = 0$. If E_+' and E_-' are the projections constructed for φ^{-1} then $E_+' + E_-' = I$. If $\varphi(\mathscr{L}E_+) \neq \mathscr{L}_1\varphi(E_+)$ then there exists an element $G_0 \in \mathscr{L}_1$ such that $G_0 \leq \varphi(E_+)$ and $G_0 \notin \varphi(\mathscr{L}E_+)$. By Lemma 2.10, $\varphi^{-1}(G_0E_+') \in \mathscr{L}$ and $\varphi^{-1}(G_0E_-') \in \mathscr{L}^\perp$. Now, if $G_0E_-' = 0$, then $\varphi^{-1}(G_0) = \varphi^{-1}(G_0E_+') \in \mathscr{L}$ and also $\varphi^{-1}(G_0) \leq E_+$; hence $\varphi^{-1}(G_0) \in \mathscr{L}E_+$, and we arrive at a contradiction. Therefore $G_0E_-' \neq 0$ and we write $G = G_0E_-'$. Then $G \leq \varphi(E_+)$ and $\varphi^{-1}(G) \in \mathscr{L}^\perp$. Thus $I - \varphi^{-1}(G) \in \mathscr{L}$ and, by Lemma 2.10, $\varphi(E_+) - G = \varphi((I - \varphi^{-1}(G))E_+) \in \mathscr{L}_1$. Since $E_+ \in \mathscr{L}\cap\mathscr{L}^\perp$ (Lemma 2.10), $\varphi(E_+) \in \mathscr{L}_1 \cap \mathscr{L}_1^\perp$. Combining these facts we get $G = \varphi(E_+)(I - (\varphi(E_+) - G)) \in \mathscr{L}_1^\perp$; hence $G \in \mathscr{L}_1 \cap \mathscr{L}_1^\perp$. Thus $\varphi^{-1}(G) \in \mathscr{L}\cap\mathscr{L}^\perp$ and, since $G_0 = G_0E_+' + G_0E_-' = G_0E_+' + G$, we have $\varphi^{-1}(G_0) = \varphi^{-1}(G_0E_+') + \varphi^{-1}(G) \in \mathscr{L}$. Since $\varphi^{-1}(G_0) \leq E_+$, $\varphi^{-1}(G_0) \in \mathscr{L}E_+$. This contradicts the choice of G_0 and proves that $\varphi(\mathscr{L}E_+) = \mathscr{L}_1\varphi(E_+)$. The fact that $\varphi(\mathscr{L}E_-) = \mathscr{L}_1^\perp\varphi(E_-)$ is proved in a similar way. □

Let E_+, E_- and P_0 be as in Proposition 2.11. We first assume that $E_+ = I$ (i.e. $E_- = P_0 = 0$). Then $\varphi(\mathscr{L}) = \mathscr{L}_1$. The following argument was used in [MT2].

Since $\mathscr{L}''$ is a commutative von Neumann algebra there is a maximal abelian selfadjoint algebra (i.e. a masa) $\mathscr{R}$ such that $\mathscr{L}'' \subseteq \mathscr{R} \subseteq \mathscr{L}' = \mathscr{A} \cap \mathscr{A}^*$. The restriction of φ to $\mathscr{R}$ is a Jordan $*$-isomorphism and therefore its image, $\varphi(\mathscr{R})$ is a abelian selfadjoint algebra, written $\mathscr{R}_1$. As φ^{-1}, restricted to $\mathscr{L}_1'$, is Jordan $*$-isomorphism of the von Neumann algebra $\mathscr{L}_1'$ onto $\mathscr{L}'$ it preserves commutativity. Thus if $T \in \mathscr{R}_1' \subseteq \mathscr{L}_1'$ then $\varphi^{-1}(T) \in \mathscr{R}' = \mathscr{R}$; hence

$T \in \mathcal{R}_1$. This shows that $\mathcal{R}_1 = \varphi(\mathcal{R})$ is a masa. Now the restriction of φ to $\mathcal{R}$ is a Jordan $*$-isomorphism of masa's and, thus, is a $*$-isomorphism. By [KR, Theorem 9.3.1] we know that φ, restricted to $\mathcal{R}$, can be implemented by some unitary operator V; i.e. $\varphi(T) = V^*TV$, $T \in \mathcal{R}$.

Since $\varphi(\mathcal{L}) = \mathcal{L}_1$, $V^*\mathcal{L}V = \mathcal{L}_1$ and $V^*\mathcal{A}V = \mathcal{A}_1$. If we now write $\psi(T) = V\varphi(T)V^*$ then ψ is a Jordan partial $*$-isomorphism from $\mathcal{A}$ onto itself leaving $\mathcal{R}$ (and, in particular, $\mathcal{L}''$) elementwise fixed.

Lemma 2.12. *Suppose $E_+ = I$ and ψ is as above. Then for every interval $E_2 - E_1$ (with E_1, $E_2 \in \mathcal{L}$ and $E_1 \leq E_2$) and $T \in \operatorname{Alg}\mathcal{L}$ we have*

$$\psi(T(E_2 - E_1)) = \psi(T)(E_2 - E_1) \quad \text{and} \quad \psi((E_2 - E_1)T) = (E_2 - E_1)\psi(T).$$

Proof. We have $TE_2(I - E_1) = (E_2 - E_1)T(E_2 - E_1) + E_1T(E_2 - E_1)$ (as $(I - E_2)T(E_2 - E_1) = 0$). Also $(E_2 - E_1)TE_1 = 0$ (as $E_1 \in \mathcal{L}$) and thus

$$\begin{aligned} &\psi(T(E_2 - E_1)) \\ &\quad = \psi((E_2 - E_1)T(E_2 - E_1)) + \psi(E_1T(E_2 - E_1) + (E_2 - E_1)TE_1) \\ &\quad = (E_2 - E_1)\psi(T)(E_2 - E_1) + E_1\psi(T)(E_2 - E_1) + (E_2 - E_1)\psi(T)E_1 \\ &\quad = (E_2 - E_1)\psi(T)(E_2 - E_1) + E_1\psi(T)(E_2 - E_1) \\ &\quad = \psi(T)(E_2 - E_1). \end{aligned}$$

The other statement is proved in a similar way. □

We shall also need the following lemma.

Lemma 2.13. *For every $N \in \mathcal{L}$ we write $G_1(N)$ for the projection onto $[N\mathcal{A}(I - N)(H)]$ and $G_2(N)$ for the projection onto $[(I - N)\mathcal{A}^*N(H)]$. Let $G_1 = \bigvee\{G_1(N) : N \in \mathcal{L}\}$ and $G_2 = \bigvee\{G_2(N) : N \in \mathcal{L}\}$. Then $G_1 \vee G_2 \geq I - P_0$. Hence if $P_0 = 0$ then $G_1 \vee G_2 = I$.*

Proof. Write $F = I - G_1 \vee G_2$. As $FG_1 = FG_2 = 0$ we have $FN\mathcal{A}(I - N) = 0$ and $F(I - N)\mathcal{A}^*N = 0$ for every $N \in \mathcal{L}$. Hence $N(F\mathcal{A})(I - N) = N(\mathcal{A}F)(I - N) = 0$ for every $N \in \mathcal{L}$. It follows that both $F\mathcal{A}$ and $\mathcal{A}F$ are contained in $\mathcal{A}^*$. As both are also contained in $\mathcal{A}$, we have $F\mathcal{A} \subseteq \mathcal{L}'$ and $\mathcal{A}F \subseteq \mathcal{L}'$ which implies that $F \leq P_0$. □

Proposition 2.14. *Assume that $E_+ = I$ and ψ is as above then ψ is multiplicative.*

Proof. From Lemma 2.13 it follows that we can find sequences $\{N_i\}_{i=1}^{\infty}$ and $\{M_i\}_{i=1}^{\infty}$ of projections in $\mathcal{L}$ such that $(\bigvee_i G_1(N_i)) \vee (\bigvee_i G_2(M_i)) = I$. We can, therefore, find a sequence of projections $\{E_i\}$ in $\mathcal{L}''$ such that:

(i) each E_i is an interval in $\mathcal{L}$. (Note that $G_1(N) \in \mathcal{L}$ and $G_2(N) \in \mathcal{L}^{\perp}$ for every $N \in \mathcal{L}$).
(ii) For $i \neq j$, $E_iE_j = 0$;
(iii) $\sum_i E_i = I$; and
(iv) For each i there is some $N \in \mathcal{L}$ such that either $E_i \leq G_1(N)$ or $E_i \leq G_2(N)$.

Now fix T, $S \in \mathcal{A}$. For $i \neq j$ we have

$$\begin{aligned} \psi(E_iTSE_j) &= \psi(E_iTSE_j + SE_jE_iT) = \psi(E_iT)\psi(SE_j) + \psi(SE_j)\psi(E_iT) \\ &= E_i\psi(T)\psi(S)E_j + \psi(S)E_jE_i\psi(T) = E_i\psi(T)\psi(S)E_j. \end{aligned}$$

(We use Lemma 2.12 to see that $\psi(E_iT) = E_i\psi(T)$ and $\psi(SE_j) = \psi(S)E_j$.) Hence

$$(*) \qquad \psi(E_iTSE_j) = E_i\psi(T)\psi(S)E_j\,, \qquad i \neq j\,.$$

Suppose E_i is such that $E_i \leq G_1(N)$ for some $N \in \mathscr{L}$. For every $R \in \mathscr{A}$ we have

$$\psi(E_iTE_iSE_iNR(I-N)) = \psi(E_iTE_iSE_iNR(I-N)) + \psi(NR(I-N)E_iSE_iTE_i)$$

(as $E_i \leq G_1(N) \leq N$). Since ψ is a Jordan isomorphism,

$$\begin{aligned}\psi(E_iTE_iSE_iNR(I-N)) &= \psi(E_iTE_i)\psi(E_iSE_i)\psi(NR(I-N)) \\ &+ \psi(NR(I-N))\psi(E_iSE_i)\psi(E_iTE_i) = E_i\psi(T)E_i\psi(S)E_iN\psi(R)(I-N) \\ &+ N\psi(R)(I-N)E_i\psi(S)E_i\psi(T)E_i = E_i\psi(T)E_i\psi(S)E_iN\psi(R)(I-N)\,.\end{aligned}$$

We get

$$\psi(E_iTE_iSE_iNR(I-N)) = E_i\psi(T)E_i\psi(S)E_iN\psi(R)(I-N)$$

and, replacing S by TE_iS and T by I we get

$$\begin{aligned}\psi(E_iTE_iSE_iNR(I-N)) &= E_i\psi(TE_iS)E_iN\psi(R)(I-N) \\ &= \psi(E_iTE_iSE_i)N\psi(R)(I-N)\end{aligned}$$

and from these two equations we conclude:

$$E_i\psi(T)E_i\psi(S)E_iN\psi(R)(I-N) = \psi(E_iTE_iSE_i)N\psi(R)(I-N)\,.$$

Since this holds for all $R \in \mathscr{A}$ and since $E_i = E_iG_1(N)$ we get (recall that $G_1(N)$ is the projection onto $[N\mathscr{A}(I-N)(H)]$),

$$(**) \qquad \psi(E_iTE_iSE_i) = E_i\psi(T)E_i\psi(S)E_i\,.$$

Using the fact that ψ is a Jordan homomorphism again we get

$$\begin{aligned}\psi(E_iTSE_i + E_iSE_iTE_i) &= E_i\psi(E_iTSE_i + SE_iE_iT)E_i \\ &= E_i(\psi(E_iT)\psi(SE_i) + \psi(SE_i)\psi(E_iT))E_i \\ &= E_i\psi(T)\psi(S)E_i + E_i\psi(S)E_i\psi(T)E_i\end{aligned}$$

and, using $(**)$ with T and S reserved, we have

$$\psi(E_iTSE_i + E_iSE_iTE_i) = E_i\psi(T)\psi(S)E_i + \psi(E_iSE_iTE_i)\,.$$

Hence

$$(***) \qquad \psi(E_iTSE_i) = E_i\psi(T)\psi(S)E_i\,.$$

This was shown to hold for E_i with the property that $E_i \leq G_1(N)$. We wish to show it also for E_i satisfying $E_i \leq G_2(N)$ for some $N \in \mathscr{L}$. For that simply replace $\mathscr{A}$ by $\mathscr{A}^*$, ψ by ψ^* (where $\psi^*(T) = \psi(T)^*$, $T \in \mathscr{A}^*$), T by S^* and S by T^* and note that $G_2(N)$ is now $G_1(I-N)$ (with $\mathscr{A}^*$ in place of $\mathscr{A}$). We now get,

$$\psi^*(E_iS^*T^*E_i) = E_i\psi^*(S^*)\psi^*(T^*)E_i$$

and, thus, by taking adjoints, $\psi(E_iTSE_i) = E_i\psi(T)\psi(S)E_i$; so that $(***)$ holds for every i.

Now we write

$$\psi(TS) = \sum_{ij} E_i\psi(TS)E_j = \sum_i E_i\psi(TS)E_i + \sum_{i\neq j} E_i\psi(TS)E_j$$

and, using $(*)$, $(***)$, we conclude

$$\psi(TS) = \sum_i E_i\psi(T)\psi(S)E_i + \sum_{i\neq j} E_i\psi(T)\psi(S)E_j = \psi(T)\psi(S). \quad \square$$

We are now ready to prove the main theorem.

Theorem 2.15. *Let $\mathscr{L}$ and $\mathscr{L}_1$ be commutative lattices of projections acting on a separable Hilbert space H. Let θ: $\mathrm{Alg}\mathscr{L} \to \mathrm{Alg}\mathscr{L}_1$ be a partial triple product isomorphism onto $\mathrm{Alg}\mathscr{L}_1$. (In particular this holds for a linear surjective isometry.) Let $U = \theta(I)$ and φ: $\mathrm{Alg}\mathscr{L} \to \mathrm{Alg}\mathscr{L}_1$ be defined by $\varphi(T) = U^*\theta(T)$, $T \in \mathrm{Alg}\mathscr{L}_1$. Then there is a projection $E \in \mathscr{L}\cap\mathscr{L}^\perp \subseteq (\mathrm{Alg}\mathscr{L})'$ such that:*

(1) *φ, restricted to $(\mathrm{Alg}\mathscr{L})E$, is an isomorphism of $(\mathrm{Alg}\mathscr{L})E$ onto $(\mathrm{Alg}\mathscr{L}_1)\varphi(E)$ mapping $\mathscr{L}E$ onto $\mathscr{L}_1\varphi(E)$.*
(2) *φ, restricted to $(\mathrm{Alg}\mathscr{L})(I - E)$, is an anti-isomorphism of $(\mathrm{Alg}\mathscr{L})(I - E)$ onto $(\mathrm{Alg}\mathscr{L}_1)\varphi(I - E)$ mapping $\mathscr{L}(I - E)$ onto $\mathscr{L}_1^\perp\varphi(I - E)$.*

Proof. Recall (Proposition 2.11) that there are projections E_+, E_- and P_0, in $\mathscr{L}\cap\mathscr{L}^\perp$, such that $I = E_+ + E_- + P_0$ and $\varphi(\mathscr{L}E_+) = \mathscr{L}_1\varphi(E_+)$, $\varphi(\mathscr{L}E_-) = \mathscr{L}_1^\perp\varphi(E_-)$ and $\mathscr{A}P_0 \subseteq \mathscr{L}'P_0$ (where $\mathscr{A} = \mathrm{Alg}\mathscr{L}$). Since we can decompose $\mathscr{A}$ as the direct sum of $\mathscr{A}E_+$, $\mathscr{A}E_-$ and $\mathscr{A}P_0$, it suffices to prove the result for the restrictions of φ to each of these algebras.

For $\mathscr{A}E_+$ we can assume $E_+ = I$ and use Proposition 2.15 (with ψ as in the discussion preceding Lemma 2.12) to conclude that ψ is multiplicative and, thus, so is φ (as $\varphi(T) = V^*\psi(T)V$ and V is unitary).

For $\mathscr{A}E_-$ we can assume that $E_- = I$, i.e. $\varphi(\mathscr{L}) \subseteq \mathscr{L}_1^\perp$. Fix an involution J of H, i.e. an isometric conjugative linear mapping J of H onto H such that $J^2 = I$. Then it is easy to check that the map $T \mapsto JT^*J$ is a $*$-anti-isomorphism of $B(H)$ onto itself. Define $\psi_0(T) = \varphi(JT^*J)$ for $T \in (\mathrm{alg}(J\mathscr{L}J))^* = \mathrm{Alg}(J\mathscr{L}J)^\perp$. This defines a Jordan partial $*$-isomorphism from $\mathrm{Alg}(J\mathscr{L}J)^\perp$ onto $\mathrm{Alg}\mathscr{L}_1$ that maps I into I and $(J\mathscr{L}J)^\perp$ into $\mathscr{L}_1$. Hence, by the part just proved above, φ_0 is an isomorphism and it follows that φ is an anti-isomorphism.

For $\mathscr{A}P_0 = \mathscr{L}'P_0$ the result follows from [K, Theorem 10]. $\square$

Corollary 2.16. *Every partial triple product isomorphism of CSL algebras is continuous.*

Proof. This follows from Theorem 2.15 and Theorem 3.1 of [GM]. (The argument used in the proof of Theorem 2.15 can be used to show that [GM, Theorem 3.1] holds for anti-isomorphisms.) $\square$

Corollary 2.17. *Let $\mathscr{L}$ and $\mathscr{L}_1$ be commutative subspace lattices and suppose that there is an isometry from $\mathrm{Alg}\mathscr{L}$ onto $\mathrm{Alg}\mathscr{L}_1$. If $\mathscr{L}$ is completely distributive, then so is $\mathscr{L}_1$ and the results of* [MT2] *can be applied.*

Proof. This follows from Theorem 2.15 and [GM, Corollary 2.2]. $\square$

Remark. Combining Theorem 2.15 (for an isometry θ) with Theorem 2.1) of [DP] provides an alternative proof of the main result of [MT2] (see Theorem 1.2).

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